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Dissipative quantum systems and the heat capacity

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We present a detailed study of the quantum dissipative dynamics of a charged particle in a magnetic field. Our focus of attention is the effect of dissipation on the low- and high-temperature behaviors of the specific heat at constant volume. After providing a brief overview of two distinct approaches to the statistical mechanics of dissipative quantum systems, viz., the ensemble approach of Gibbs and the quantum Brownian motion approach due to Einstein, we present exact analyses of the specific heat. While the low-temperature expressions for the specific heat, based on the two approaches, are in conformity with power-law temperature dependence, predicted by the third law of thermodynamics, and the high-temperature expressions are in agreement with the classical equipartition theorem, there are surprising differences between the dependencies of the specific heat on different parameters in the theory, when calculations are done from these two distinct methods. In particular, we find puzzling influences of boundary confinement and the bath-induced spectral cutoff frequency. Further, when it comes to the issue of approach to equilibrium, based on the Einstein method, the way the asymptotic limit ($t \to \infty$) is taken seems to assume significance.

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I. INTRODUCTION

Recent years have seen great strides in the statistical mechanics of dissipative quantum systems [1]. Dissipation arises when the quantum degrees of freedom of a heat bath, which is coupled to a subsystem of interest, are projected (or integrated) out of the Hilbert space of the total system. Two different approaches, detailed below in Sec. II, have been used in this context: (i) the usual Gibbs approach that focuses on the partition function [2] and (ii) the Einstein approach that hinges on a quantum Langevin equation for the subsystem [3]. Lately it has been argued that the presence of quantum dissipation yields a satisfactory behavior of the fundamental thermodynamic attribute, viz., the heat capacity, as far as the low-temperature properties are concerned [4–7]. Here we will point out that there are some puzzling issues even for the high-temperature limit of the heat capacity, apart from the intriguing low-temperature attributes. Before we address this question, it is important to review the kind of subsystem we have in mind and the foundational basis of statistical mechanics, which we do below. While our present discussion as well as that in Sec. II are set within the domain of classical statistical mechanics, extension to quantum mechanics can be easily carried out, as indicated in Sec. III. But we want to first concentrate on some preliminaries about the subject of statistical mechanics itself.

Statistical mechanics provides the microscopic basis of the macroscopic properties of a system described by the subject of thermodynamics. Though the power of statistical mechanics comes to the fore in its full glory for an interacting many-body system, such as in the exact formulation of second-order phase transitions by means of the two-dimensional Ising model [8], many of the intricacies can be elucidated for just a single entity, albeit in contact with a heat bath comprising an infinitely large number of (invisible) degrees of freedom. It is this simplified approach to statistical mechanics in the context of a single particle embedded in a heat bath that we shall adopt in this paper.

The dynamics of a particle of mass m is described by the system Hamiltonian defined by

$$\mathcal{H}_S = \frac{\vec{p}^2}{2m} + V(\vec{q},\vec{p}), \tag{1}$$

where $\vec{p}$ is the canonical momentum vector of the particle moving under an arbitrary potential $V(\vec{q},\vec{p})$ which in general is a function of the generalized coordinate vector $\vec{q}$ and the generalized momentum $\vec{p}$. We shall discuss three distinct cases in the sequel:

(a) *free particle*,

$$V(\vec{q},\vec{p}) = 0, \tag{2}$$

(b) *Harmonic oscillator*,

$$V(\vec{q},\vec{p}) = \frac{1}{2}m\omega_0^2\vec{q}^2, \tag{3}$$

with ω_0 being the frequency of the oscillator, and

(c) *charged oscillator in a magnetic field*, that is described by a momentum and coordinate-dependent potential,

$$V(\vec{q},\vec{p}) = -\frac{e}{2mc}(\vec{p}\,.\,\vec{A}(\vec{q}) + \vec{A}(\vec{q})\,.\,\vec{p}) + \frac{e^2}{2mc^2}\vec{A}^2(\vec{q}) + \frac{1}{2}m\omega_0^2\vec{q}^2, \tag{4}$$

with $\vec{A}(\vec{q})$ being the vector potential, the curl of which yields the magnetic field $\vec{B}$,

$$\vec{B} = \vec{\nabla} \times \vec{A}(\vec{q}). \tag{5}$$

It is evident that for zero vector potential, case (c) reduces to (b). If additionally, ω_0 is also zero, case (a) is obtained. In what way are these limiting situations arrived at, for a quantum dissipative system, will indeed be the focus of our discussion below.

It should be mentioned here that the problem of a charged oscillator in a magnetic field is relevant in the context of Landau diamagnetism [9] which has had a deep impact on modern condensed-matter physics through phenomena such as the quantum Hall effect [10]. Landau diamagnetism, which is purely quantum in origin, is characterized by strong boundary effects that can be mimicked by the oscillator potential [11]. The presence of a quantum bath, comprising say, bosonic excitations like phonons, lends additional richness to the problem as it allows us to study the effect of dissipation on Landau diamagnetism [12]. In this paper however our focus of attention is mainly on the thermodynamic property of the heat capacity, but we also make some remarks on diamagnetism.

While in classical mechanics the trajectory of a particle in the phase space is determined, once the initial values of $\vec{q}$ and $\vec{p}$ are given, the point of statistical mechanics is that the phase trajectory randomly changes from one "realization" of the system to another. It is this multitude of trajectories corresponding to multiple realizations of the system that yields the concept of "ensemble" in statistical mechanics. An ensemble means a collection of possible realizations of the system. Thermal equilibrium is said to be reached when experiments are repeated so many times that all possible trajectories (realizations) in the phase space are explored and this yields the notion of "mixing" [13]. Evidently, the state of thermal equilibrium is the one in which transients have died out and hence ensemble averages become time independent.

With these preliminaries, the outline of the paper is as follows. In Sec. II, we review the Gibbs and Einstein approaches to statistical mechanics. Although our treatments are couched in classical terms similar results hold for quantum phenomena as well. With these approaches in the background we summarize in Sec. III, the newly developed subject of dissipative quantum systems. In Sec. IV we analyze the results for the heat capacity for the three problems (a–c) and point out certain surprises when we consider the various limits of case (c). In Sec. V, we summarize the results.

II. GIBBS AND EINSTEIN APPROACHES TO STATISTICAL MECHANICS

The remarkable thesis of Gibbs is that for a system in thermal equilibrium the observed properties of the system can be computed from a weighted average of the values of the relevant observable at all possible phase points that lie on a constant time slice. The ensemble average of an observable $X(\vec{q},\vec{p})$ in equilibrium (indicated by the subscript "eq" below) is defined by

$$\langle X(\vec{q},\vec{p})\rangle_{eq} = \mathrm{Tr}[\rho(\vec{q},\vec{p})X(\vec{q},\vec{p})], \tag{6}$$

where "Tr" (trace) implies an integration over the entire phase space in classical statistical mechanics, whereas it is a sum over possible eigenstates of the full $\mathcal{H}_S$ in Eq. (1) in quantum statistical mechanics. The Gibbs-Boltzmann weight function $\rho(\vec{q},\vec{p})$ is what is called a density matrix, given by

$$\rho(\vec{q},\vec{p}) = \frac{\exp[-\beta\mathcal{H}_S(\vec{q},\vec{p})]}{\mathcal{Z}_S}, \tag{7}$$

where $\beta(=(k_BT)^{-1})$ is the inverse temperature, k_B being the Boltzmann constant. The normalization factor $\mathcal{Z}_S$, referred to as the partition function:

$$\mathcal{Z}_S = \mathrm{Tr}\{\exp[-\beta\mathcal{H}_S(\vec{q},\vec{p})]\}, \tag{8}$$

provides the critical link between statistical mechanics and thermodynamics as it leads to the Helmholtz free energy $\mathcal{F}$ through the relation

$$\mathcal{F}_S = -\frac{1}{\beta}\ln\,\mathcal{Z}_S. \tag{9}$$

From $\mathcal{F}_S$ all thermodynamic properties can be derived.

It is of course outside the realm of Gibbsian statistical mechanics to address the issue of how equilibrium is reached. That question has to be posed in terms of models of nonequilibrium statistical mechanics, which are however not as robust and time tested as the formulation of equilibrium statistical mechanics encapsulated by Eqs. (7)–(9). One model that stands out in this regard is based on the idea of Brownian motion [14]. In the latter one imagines the particle (much like the pollen particle of Brown [15]), the Hamiltonian of which is given by Eq. (1), is in contact with a heat bath that drives stochastic (noisy) fluctuations into the system. The idea of Brownian motion is very physical in that if one tags the particle by taking camera snapshots at different times, its dynamics would indeed appear to be random, when the particle is out of equilibrium, and even when it is in equilibrium. The stochastic dynamics is captured by the time-dependent distribution function $\mathcal{P}(\vec{q},\vec{p},t)$ in phase space that obeys the Fokker-Planck-Smoluchowski-Kramers equation [16],

$$\frac{\partial}{\partial t}\mathcal{P}(\vec{q},\vec{p},t) = \Bigg\{-\frac{\vec{p}}{m}.\vec{\nabla}_q + \vec{\nabla}_p.(\vec{\nabla}_qV(\vec{q}) + \gamma\vec{p}) + m\gamma k_BT\nabla_p^2\Bigg\}\mathcal{P}(\vec{q},\vec{p},t), \tag{10}$$

where γ is the friction constant. The quantity $\mathcal{P}$ plays the same role in nonequilibrium as ρ does in equilibrium. Thus the averaged time evolution of the dynamical variable $X(\vec{q},\vec{p})$ is given by

$$\bar{X}(t) = \int d\vec{q}d\vec{p}X(\vec{q},\vec{p})\mathcal{P}(\vec{q},\vec{p},t). \tag{11}$$

With the temperature-dependent prefactor in front of ∇^2, it is ensured that the stationary state is indeed the thermal equilibrium state, described by ρ in Eq. (7). This is consistent with the fluctuation-dissipation theorem.

It is pertinent to mention here that the time-dependent approach, as formulated through Eq. (10), is based on what is called the "Schrödinger picture." An equivalent description obtains through the "Heisenberg picture" in which one directly considers the dynamical equations of motion,

$$\frac{\partial\vec{q}}{\partial t} = \frac{\vec{p}}{m},$$

$$\frac{\partial\vec{p}}{\partial t} = -m\omega_0^2\vec{q} - \frac{e}{c}(\vec{q}\times\vec{B}) - \gamma\vec{p}(t) + \vec{f}(t). \tag{12}$$

The set of Eq. (12) is called the Langevin equation in which the force $\vec{f}(t)$ is a stochastic noise, with

$$\langle \vec{f}(t) \rangle = 0$$

$$\langle f_\mu(t) f_\nu(t') \rangle = 2m\gamma k_B T \delta(t-t') \delta_{\mu\nu}, \quad \mu,\nu = x,y,z. \quad (13)$$

The Brownian motion model, as formulated through Eqs. (12) and (13), is appropriately dubbed the "Einstein approach to statistical mechanics" [17].

III. DISSIPATIVE QUANTUM SYSTEMS

In this section we move from the classical to the quantal domain and consider the case in which the quantum subsystem is put into contact with a heat bath that is also quantum mechanical. Before we indicate the steps necessary for Brownian motion in terms of what is referred to as the quantum Langevin equation [3], it is useful to backtrack and indicate how classical Langevin Eq. (12) itself is derived from a system-plus-bath method. Here we start from a treatment of Zwanzig [18] in which the Hamiltonian in Eq. (1) is extended as

$$\mathcal{H} = \mathcal{H}_S + \sum_j \left[\frac{\vec{p}_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 \left(\vec{q}_j - \frac{C_j \vec{q}}{m_j \omega_j^2} \right)^2 \right]. \quad (14)$$

Upon expanding the square over the round brackets it is evident that the Hamiltonian contains a linear coupling between the coordinate $\vec{q}$ of the subsystem and the coordinate $\vec{q}_j$ of the harmonic bath with C_j being a coupling constant.

From Eq. (14) it is easy to write down Hamilton's equations of motion, solve for the bath coordinates and momenta, put the solutions back in the equations of motion for the subsystem variables and derive for the momentum the generalized Langevin equation [18],

$$m\ddot{\vec{q}} = -m\omega_0^2 \vec{q} - \frac{e}{c}(\dot{\vec{q}} \times \vec{B}) - m \int_0^t dt' \dot{\vec{q}}(t') \gamma(t-t') + \vec{f}(t), \quad (15)$$

where the "friction" $\gamma(t)$, that appears as a memory function, depends quadratically on C_j and the noise $\vec{f}(t)$ depends explicitly on initial coordinates and the momenta of the bath oscillators,

$$\gamma(t) = \sum_j \frac{C_j^2}{m_j \omega_j^2} \cos(\omega_j t) \quad (16)$$

$$\vec{f}(t) = \sum_j \left\{ C_j \left[\vec{q}_j(0) - \frac{C_j \vec{q}(0)}{m_j \omega_j^2} \right] \cos(\omega_j t) + \frac{C_j \vec{p}_j(0)}{m_j \omega_j} \sin(\omega_j t) \right\}. \quad (17)$$

Suffice it to note that Eq. (15) is exact and devoid of any assumption except that we have decided to integrate the equation of motion in the forward direction of time, thereby giving a sense to the "arrow of time." The next step however is a crucial one of introducing irreversibility by considering an initial ensemble of states, *a′ la* Gibbs, in which the bath variables are drawn at random from a conditional equilibrium distribution in which the initial values of the coordinates and the momenta of the Brownian particle are fixed. In particular, if the initial distribution is taken to be the canonical one given by Eq. (7), we have

$$\langle f_\mu(t) f_\nu(t') \rangle = \delta_{\mu\nu} 2mk_B T \gamma(t-t'). \quad (18)$$

The final step is to go to the limit of an infinitely large system in order to endow the harmonic-oscillator system the attribute of a heat bath. Thus

$$\frac{1}{N}\sum_j C_j^2 \ldots \to \int d\omega g(\omega), \quad m_j = m, \quad C_j = \frac{C}{\sqrt{N}}, \quad (19)$$

where $g(\omega)$ is the "spectral density." Eq. (16) then yields

$$\gamma(t) = \frac{C^2}{m} \int_0^\infty d\omega \frac{g(\omega)}{\omega^2} \cos(\omega t). \quad (20)$$

A commonly assumed form of $g(\omega)$ is the one which yields what is called Ohmic dissipation and is given by

$$g(\omega) = \frac{\omega^2}{\bar{\omega}^3}, \quad \omega < \bar{\omega}$$

$$= 0, \quad \omega > \bar{\omega}, \quad (21)$$

$\bar{\omega}$ being a high-frequency cutoff. Employing Eq. (21) we derive Eq. (12), implying that Ohmic dissipation corresponds to *constant* friction γ because the generalized friction coefficient reduces to $\gamma\delta(t-t')$, wherein γ equals $3\pi C^2/2m\bar{\omega}^3$ [16].

The discussion in the quantum case proceeds along similar lines in which one has to however keep track of the fact that $\vec{q}$ and $\vec{p}$ are noncommuting operators, and consequently, the noise $\vec{f}$ in Eq. (17) is also a quantum operator [3]. Additionally, because the bath oscillators are to be treated quantum mechanically, the noise correlations are not "*white*," as in Eq. (13) but are characterized by both a symmetric combination and a commutator structure, respectively, given by [16]

$$\langle \{ f_\mu(t), f_\nu(t') \} \rangle = \delta_{\mu\nu} \frac{2}{\pi} \int_0^\infty d\omega \Re[\vec{f}(\omega + i0^+)] \times \omega \coth\left(\frac{\beta\omega}{2}\right) \cos[\omega(t-t')]. \quad (22)$$

$$\langle [f_\mu(t), f_\nu(t')] \rangle = \delta_{\mu\nu} \frac{2}{i\pi} \int_0^\infty d\omega \Re[\vec{f}(\omega + i0^+)] \omega \sin[\omega(t-t')]. \quad (23)$$

IV. HEAT CAPACITY

The heat capacity or the specific heat at constant volume is the most basic thermodynamic property. In equilibrium thermodynamics, it is defined by [19]

$$C = -k_B \beta^2 \left(\frac{\partial U}{\partial \beta} \right)_V, \tag{24}$$

where the internal energy U is calculated within the Gibbs approach from the reduced partition function ($\mathcal{Z}_R$), by employing the relation

$$U = -\frac{\partial}{\partial \beta} \ln \mathcal{Z}_R \tag{25}$$

where [1,6]

$$\mathcal{Z}_R = \frac{[\mathrm{Tr}_{S+B}(e^{-\beta \mathcal{H}})]}{[\mathrm{Tr}_B(e^{-\beta \mathcal{H}_B})]}, \tag{26}$$

$\mathcal{H}$ being the full Hamiltonian as in Eq. (14), $\mathrm{Tr}_{S+B}(\ldots)$ represents a trace over the system and bath, $\mathcal{H}_B$ is the Hamiltonian of the bath, $\mathrm{Tr}_B(\ldots)$ represents a trace over the bath only and β defines the temperature of the bath in which the composite many-body system, described by the full Hamiltonian $\mathcal{H}$, is embedded. It is customary to rewrite $\mathcal{Z}$ as a functional integral [20],

$$\mathcal{Z}_{\mathcal{R}} = \oint \mathcal{D}[\vec{q}, \vec{p}, \vec{q_j}, \vec{p_j}] \exp\left(-\frac{1}{\hbar} \mathcal{A}_e[\vec{q}, \vec{p}, \vec{q_j}, \vec{p_j}] \right), \tag{27}$$

where $\hbar$ is the Planck constant and $\mathcal{A}_e$ is the so-called Euclidean action, defined by

$$\mathcal{A}_e = \int_0^{\hbar\beta} d\tau \mathcal{L}(\tau), \tag{28}$$

$\mathcal{L}(\tau)$ being the Lagrangian written in terms of the "imaginary time" $\tau(=i\hbar\beta)$.

On the other hand, in the Einstein approach, we evaluate U directly from the long time limit of $\langle \mathcal{H}_S^{eff}(t) \rangle$, where the superscript "eff" denotes "effective," in the sense described below in Sec. IV C, the latter being calculable from the quantum Langevin equation. The specific heat is then calculated using Eq. (24). This leads to unambiguous results for the specific heat as shown below. We illustrate in Sec. IV below the application of Gibbs and Einstein approaches to the calculation of the heat capacity for the charged oscillator in a magnetic field.

A. Gibbs approach ($\omega_0 \neq 0$)

Before we discuss the calculation of C^{Gibbs} for the dissipative charged oscillator in a magnetic field it is useful to indicate the steps for the simpler problem without dissipative coupling, viz, that described by $\mathcal{H}_S$ alone [Eqs. (1) and (4)] [7]. The corresponding Lagrangian for the two-dimensional motion in the plane normal to the field is given by

$$\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} m\omega_0^2 (x^2 + y^2) - \frac{e}{c}(\dot{x}A_x + \dot{y}A_y). \tag{29}$$

It is customary to work in the so-called "symmetric gauge" in which

$$A_x = -\frac{1}{2} yB, \quad A_y = \frac{1}{2} xB. \tag{30}$$

The Euclidean action can be written as

$$\mathcal{A}_e[x,y] = \frac{m}{2} \int_0^{\hbar\beta} d\tau [(\dot{x}(\tau)^2 + \dot{y}(\tau)^2) + \omega_0^2 (x(\tau)^2 + y(\tau)^2) - i\omega_c (x(\tau)\dot{y}(\tau) - y(\tau)\dot{x}(\tau))], \tag{31}$$

ω_c being the "cyclotron frequency" given by

$$\omega_c = \frac{eB}{mc}. \tag{32}$$

Introducing

$$x(\tau) = \sum_j \tilde{x}(\nu_j) \exp(-i\nu_j \tau), \tag{33}$$

where ν_j's are the so-called Matsubara frequencies, defined by

$$\nu_j = \frac{2\pi j}{\hbar \beta} \quad j = 0, \pm 1, \pm 2, \ldots, \tag{34}$$

we find

$$\mathcal{A}_e[z_+, z_-] = \frac{1}{2} m\hbar\beta \sum_{j=-\infty}^{\infty} [(\nu_j^2 + \omega_0^2 + i\omega_c \nu_j) \tilde{z}_+^*(\nu_j) \tilde{z}_+(\nu_j) + (\nu_j^2 + \omega_0^2 - i\omega_c \nu_j) \tilde{z}_-^*(\nu_j) \tilde{z}_-(\nu_j)], \tag{35}$$

where

$$\tilde{z}_\pm(\nu_j) = \frac{1}{\sqrt{2}} (\tilde{x}(\nu_j) \pm i\tilde{y}(\nu_j)). \tag{36}$$

As shown in Ref. [7] the partition function $\mathcal{Z}_S$ in Eq. (8) can be written as [cf., also Eq. (27)]

$$\mathcal{Z}_S = \prod_{j=1}^{\infty} \mathcal{Z}_j^+ \mathcal{Z}_j^-, \tag{37}$$

where,

$$\mathcal{Z}_j^+ = \frac{1}{\sqrt{2\pi\hbar^2\beta/m}} \int_{-\infty}^{\infty} dz_+(0) \exp\left[-\frac{m\beta\omega_0^2}{2} |z_+(0)|^2 \right] \times \prod_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d \,\mathrm{Re}\, z_+ d \,\mathrm{Im}\, z_+}{\pi/(m\beta\nu_j^2)} \times \exp[-m\beta(\nu_j^2 + \omega_0^2 - i\omega_c \nu_j)(\mathrm{Re}\, z_+^2 + \mathrm{Im}\, z_+^2)], \tag{38}$$

and

$$\mathcal{Z}_j^- = (\mathcal{Z}_j^+)^*. \tag{39}$$

Carrying out the Gaussian integrals we find

$$\mathcal{Z}_j^+ = \frac{1}{\beta\hbar\omega_0} \frac{\nu_j^2}{(\nu_j^2 + \omega_0^2 - i\omega_c \nu_j)}. \tag{40}$$

Hence,

$$\mathcal{Z}_S = \left(\frac{1}{\beta\hbar\omega_0}\right)^2 \prod_{j=1}^{\infty} \frac{\nu_j^4}{(\nu_j^2+\omega_0^2)^2+\omega_c^2\nu_j^2}. \tag{41}$$

Turning now to the dissipative system described by the full many-body Hamiltonian in Eq. (14) we can similarly derive [7]

$$\mathcal{Z}_{\mathcal{R}}(\omega_0) = \frac{1}{(\hbar\beta\omega_0)^2} \prod_{j=1}^{\infty} \frac{\nu_j^4}{(\nu_j^2+\omega_0^2+\nu_j\tilde{\gamma}(\nu_j))^2+\omega_c^2\nu_j^2}, \tag{42}$$

where $\tilde{\gamma}(\nu_j)$ is the frequency-dependent (i.e., ν_j) friction coefficient. The Ohmic dissipation model, discussed earlier in Eq. (21) that yields constant friction, is not suitable for calculating $\mathcal{Z}$ as it leads to a singularity. In order to regularize the latter it is convenient to introduce a "Drude cutoff" by writing the spectral density as [cf., Eq. (21)]

$$g(\omega) = \frac{2m\gamma}{\pi C^2} \frac{\omega^2}{1+\dfrac{\omega^2}{\omega_D^2}}. \tag{43}$$

Correspondingly,

$$\tilde{\gamma}(\nu_j) = \frac{\gamma\omega_D}{(\nu_j+\omega_D)}, \quad \nu_j = \frac{2\pi j}{\hbar\beta}. \tag{44}$$

All our results in the sequel are restricted to Ohmic-Drude spectral density [Eq. (43)], though it is known that other forms of frequency dependence of the spectral density yield diverse forms of power-law dependence of the specific heat at low temperatures [21].

Inserting this form of the friction coefficient in Eq. (42) the internal energy U can be calculated as

$$U(\omega_0) = -\frac{2}{\beta} - \frac{1}{\beta}\sum_{j=1}^{3}\left[\frac{\lambda_j}{\nu}\psi\left(\frac{\lambda_j}{\nu}\right) + \frac{\lambda_j'}{\nu}\psi\left(\frac{\lambda_j'}{\nu}\right)\right] + \frac{2}{\beta}\frac{\omega_D}{\nu}\psi\left(\frac{\omega_D}{\nu}\right), \tag{45}$$

where $\psi(z)$ is the digamma function and the arguments are

$$\lambda_1+\lambda_2+\lambda_3 = \omega_D + i\omega_c,$$

$$\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1 = \omega_0^2+\gamma\omega_D+i\omega_c\omega_D,$$

$$\lambda_1\lambda_2\lambda_3 = \omega_0^2\omega_D. \tag{46}$$

The corresponding primed λ's are obtained from the complex conjugate of Eq. (46). Finally, it is easy to derive for the heat capacity, the expression [cf., Eq. (24)] [7]

$$C^{\text{Gibbs}}_{(\omega_0\neq 0)} = -2k_B + k_B\sum_{k=1}^{3}\left\{\left(\frac{\lambda_k}{\nu}\right)^2\psi'\left(\frac{\lambda_k}{\nu}\right) + \left(\frac{\lambda_k'}{\nu}\right)^2\psi'\left(\frac{\lambda_k'}{\nu}\right)\right\} - 2k_B\left(\frac{\omega_D}{\nu}\right)^2\psi'\left(\frac{\omega_D}{\nu}\right). \tag{47}$$

We are now ready to discuss the low and high-temperature limits of the heat capacity.

(a) *Low-T limit*,

$$C^{\text{Gibbs}}_{(\omega_0\neq 0)} = \frac{2\pi}{3}\frac{\gamma}{\omega_0^2}\frac{k_B^2 T}{\hbar} + \alpha_1^G T^3 + \mathcal{O}(T^5) \tag{48}$$

where

$$\alpha_1^G = \frac{8\pi^3}{15}\frac{\gamma}{\omega_0}\frac{k_B^4}{(\hbar\omega_0)^3} \times\left\{\frac{3(\omega_c^2+\omega_0^2)}{\omega_0^2} - \left(\frac{\gamma}{\omega_0}\right)^2 - \frac{3\omega_0}{\omega_D}\left(\frac{\gamma}{\omega_0}+\frac{\omega_0}{\omega_D}\right)\right\}$$

Curiously, to leading order, the presence of the magnetic field through the cyclotron frequency disappears from $C^{\text{Gibbs}}_{(\omega_0\neq 0)}$, the expression of which matches with that of a two-dimensional quantum oscillator (Einstein oscillator). The result in Eq. (48) has been much in discussion in recent times, in the context of the third law of thermodynamics as it provides a satisfactory power-law behavior in temperature [4].

(b) *High-T limit*

At high temperatures $(\hbar\omega_c, \hbar\omega_0, \hbar\gamma, \hbar\omega_D \ll k_B T)$ our quantum system is expected to be described by classical statistical mechanics. We find

$$C^{\text{Gibbs}}_{(\omega_0\neq 0)} = 2k_B - \frac{\alpha_2^G}{T^2}. \tag{49}$$

where

$$\alpha_2^G = \frac{\hbar^2}{12k_B}(\omega_c^2+2\omega_0^2+2\gamma\omega_D)$$

In the limit of infinite temperature, therefore, we recover the expected "equipartition" result,

$$C^{\text{Gibbs}}_{(\omega_0\neq 0)} = 2k_B, \tag{50}$$

where the factor of 2 comes from two dimensions, each of which contributes k_B to the specific heat, $\frac{1}{2}k_B$ arising from the kinetic energy while the other half from the potential energy.

B. Gibbs approach (without confinement)

While studying dissipative Landau diamagnetism we have learnt that taking $\omega_0=0$ at the outset yields a puzzlingly different result from keeping ω_0 fixed, evaluating the partition function, calculating its derivatives and then setting $\omega_0=0$ [12]. It is already evident from the low-temperature specific heat [Eq. (48)] that it is not meaningful to take the limit of $\omega_0=0$ without "fixing" the coupling with the heat bath characterized by the friction coefficient γ. It is therefore of interest to take a relook at the heat-capacity calculation by investigating afresh the partition function for a charge in a magnetic field (without the oscillator potential). In this case only two roots λ_1 and λ_2 [cf., Eq. (46)] matter [7] and we find

$$\mathcal{Z}_R(F) = \frac{Nm\beta}{8\pi^3}(\gamma^2+\omega_c^2)\frac{\prod_{k=1}^{2}\Gamma\left(\frac{\lambda_k}{\nu}\right)\Gamma\left(\frac{\lambda_k'}{\nu}\right)}{\left[\Gamma\left(\frac{\omega_D}{\nu}\right)\right]^2}, \tag{51}$$

where $Z_R(F)$ is the reduced partition function of the unconfined system. The heat capacity of the unconfined system $C_{(F)}^{\text{Gibbs}}$ is then given by

$$C_{(F)}^{\text{Gibbs}} = -k_B + k_B\sum_{k=1}^{2}\left\{\left(\frac{\lambda_k}{\nu}\right)^2\psi'\left(\frac{\lambda_k}{\nu}\right) + \left(\frac{\lambda_k'}{\nu}\right)^2\psi'\left(\frac{\lambda_k'}{\nu}\right)\right\} - 2k_B\left(\frac{\omega_D}{\nu}\right)^2\psi'\left(\frac{\omega_D}{\nu}\right). \tag{52}$$

We now discuss the low and high-temperature limits of Eq. (52).

(a) *Low-T limit.*

Using asymptotic expansions as before, we find

$$C_{(F)}^{\text{Gibbs}} = \frac{2\pi}{3}\frac{\gamma}{\hbar}\frac{\left(1-\frac{\gamma}{\omega_D}\right)}{(\gamma^2+\omega_c^2)}k_B^2T - (\alpha_3^G-\alpha_4^G)T^3 + O(T^5). \tag{53}$$

where

$$\alpha_3^G = \frac{8\pi^3}{15}\frac{k_B^4}{\hbar^3\sqrt{(\gamma^2+\omega_c^2)^3}}\left\{\frac{(\gamma^3-3\gamma\omega_c^2)}{\sqrt{(\gamma^2+\omega_c^2)^3}}\left(1-\frac{3\gamma}{\omega_D}\right) + \frac{(\omega_c^3-3\omega_c\gamma^2)}{\sqrt{(\gamma^2+\omega_c^2)^3}}\left(\left(\frac{\omega_c}{\omega_D}\right)^3 + 3\gamma\frac{\omega_c}{\omega_D^2}\right)\right\},$$

$$\alpha_4^G = \frac{8\pi^3}{15}\frac{k_B^4}{(\hbar\omega_D)^3}.$$

While Eq. (53) is in conformity with the third law of thermodynamics with identical linear temperature dependence as in the case of $\omega_0 \neq 0$, it is free from the singularity issue in Eq. (48) (for $\omega_0=0$). It leads, in the limit of $\omega_D=\infty$ (infinite Drude cutoff) to the result

$$C_{(F)}^{\text{Gibbs}} = \frac{2\pi}{3\hbar}k_B^2T\frac{\gamma}{\gamma^2+\omega_c^2}. \tag{54}$$

Further, for very strong magnetic fields ($\gamma \ll \omega_c$),

$$C_{(F)}^{\text{Gibbs}} = \frac{2\pi}{3}\frac{\gamma}{\omega_c^2}\frac{k_B^2T}{\hbar}, \tag{55}$$

a harmonic-oscillator-like result with the cyclotron frequency ω_c replacing ω_0. On the other hand, for weak magnetic fields ($\gamma \gg \omega_c$),

$$C_{(F)}^{\text{Gibbs}} = \frac{2\pi}{3}\frac{k_B^2T}{\hbar}\frac{1}{\gamma}, \tag{56}$$

the free particle result in which the friction coefficient γ appears in the denominator, in agreement to the corresponding result given in [6], after a proper counting of the degree of freedom.

(b) *High-T limit.*

We find

$$C_{(F)}^{\text{Gibbs}} = k_B - \frac{\hbar^2}{12k_BT^2}(\omega_c^2+2\gamma\omega_D). \tag{57}$$

Again, equipartition theorem for a free particle (in two dimensions) prevails at $T=\infty$.

Thus the classical limit of the Landau problem, as far as the heat capacity is concerned, is that of free particle whereas an additional (parabolic) constraining potential yields harmonic-oscillator behavior.

C. Einstein approach ($\omega_0 \neq 0$)

We will now focus on the Einstein approach based on the Langevin Eq. (15) which can be recast into the following convenient form [12]:

$$\ddot{z} + \int_0^t dt'\,\bar{\gamma}(t-t')\dot{z}(t') + \omega_0^2 z = \frac{F(t)}{m}, \tag{58}$$

where

$$z = x+iy, \quad F = f_x + if_y, \quad \text{and} \quad \bar{\gamma}(t) = \gamma(t) + i\omega_c. \tag{59}$$

In order to find the time-dependent specific heat we need the internal energy which is the statistical average of some system Hamiltonian and the question is: which one? Note that the quantum Langevin Eq. (58) is derived, *a′ la* Zwanzig [18] and Ford *et al.* [3], from a "first-principles" Hamiltonian given in Eq. (14). The concomitant equations of motion for the quantum operators $\vec{q},\vec{p}$ of the system, after projecting (or integrating) out the operators $\vec{q}_j$ and $\vec{p}_j$ of the environment, yield Eq. (58). To that end, the first term in Eq. (14) given by $\mathcal{H}_S$ gets "dressed" by the environment, aided by the friction γ and the quantum noise $F(t)$. What ensues therefore is an *effective stochastic* Hamiltonian that can be constructed as follows:

$$\mathcal{H}_S^{\text{eff}} = \frac{1}{2}m\dot{z}\dot{z}^\dagger - \frac{1}{2}\hbar\omega_c + \frac{1}{2}m\omega_0^2 zz^\dagger. \tag{60}$$

Note that while right-hand side of Eq. (60) is a rewritten version of Eq. (1) [in conjunction with Eq. (4)], it is now endowed with not only an explicit time dependence but also a dependence on bath parameters [cf. Eq. (58)]. The stratagem is to first find the average of Eq. (60) over the quantum noise and identify the internal energy E as $Lim_{t\to\infty}\langle\mathcal{H}_S^{eff}(t)\rangle$, as we do in Eq. (69) below. Our claim, borne out by the results below, is that E is equivalent to the internal energy U obtained in the Gibbs approach from the reduced partition function [cf., Eq. (25)]. We need the equal-time-correlation functions

$$\zeta_1(t) = \langle z(t)z^\dagger(t)\rangle, \tag{61a}$$

$$\zeta_2(t) = \langle \dot{z}(t)\dot{z}^\dagger(t)\rangle. \tag{61b}$$

The correlation functions in Eq. (61) can be found from the analytic continuation to $t'=t$ of the unequal-time-correlation functions, e.g.,

$$\zeta_1(t,t') = \langle z(t) z^\dagger(t')\rangle, \tag{62}$$

where $z(t)$ can be further expressed in terms of the response function $\chi(t)$ as

$$z(t) = \int_0^t d\tau \chi(t-\tau)\frac{F(\tau)}{m}. \tag{63}$$

The latter is the inverse Fourier transform of $\chi(\omega)$ that can be easily written from Eq. (58) as

$$\chi(\omega) = \frac{1}{2\pi}\frac{1}{(-\omega^2 - i\omega\bar{\gamma} + \omega_0^2)}, \tag{64}$$

with

$$\bar{\gamma}(\omega) = i\omega_c + \gamma(\omega) = i\omega_c + \gamma\frac{\omega_D}{\omega_D - i\omega}. \tag{65}$$

From Eq. (62),

$$\zeta_1(t,t') = \int_0^t d\tau \int_0^{t'} d\tau' \chi(t-\tau)\chi^*(t'-\tau')\frac{\langle F(\tau)F^\dagger(\tau')\rangle}{m}, \tag{66}$$

where the force-force correlation function is given by [12,16]

$$\langle F(\tau)F^\dagger(\tau')\rangle = \int_{-\infty}^{+\infty} d\tilde{\omega} f(\tilde{\omega}) e^{-i\omega(\tau-\tau')}, \tag{67}$$

with

$$f(\tilde{\omega}) = \frac{m}{\pi}\frac{\gamma\omega_D^2}{(\omega_D^2+\tilde{\omega}^2)}\hbar\tilde{\omega}\left[\coth\left(\frac{\hbar\tilde{\omega}}{2kT}\right) - 1\right]. \tag{68}$$

Our strategy is to first calculate $\zeta_1(t,t')$ and $\zeta_2(t,t')$ (for details, see Appendix A), then set $t=t'$ and finally, in order to extract the thermal equilibrium internal energy E, take the limit $t=\infty$. We find

$$\begin{aligned} E &= \lim_{t\to\infty}\langle \mathcal{H}_S^{eff}\rangle = -\frac{1}{2}\hbar\omega_c + \frac{1}{2}m\omega_0^2 \lim_{t\to\infty}\zeta_1(t) + \frac{1}{2}m \lim_{t\to\infty}\zeta_2(t) \\ &= 2k_BT + \frac{\hbar}{2\pi}\sum_{j=1}^{3}\left\{\psi\left(1+\frac{\lambda_j}{\nu}\right)[2\omega_0^2 q_j + p_j] \right. \\ &\quad \left. + \psi\left(1+\frac{\lambda_j'}{\nu}\right)[2\omega_0^2 q_j' + p_j']\right\}, \end{aligned} \tag{69}$$

where

$$q_j = \frac{(\lambda_j - \omega_D)}{\prod_{j'}'(\lambda_j - \lambda_{j'})}, \tag{70a}$$

$$p_j = \frac{\lambda_j[\gamma\omega_D - i\omega_c(\lambda_j - \omega_D)]}{\prod_{j'}'(\lambda_j - \lambda_{j'})}. \tag{70b}$$

In the denominators of Eq. (70), the notation $\Pi_{j'}'$ implies that the $j=j'$ terms are excluded from the product. The quantities q_j' and p_j' are obtained by priming the λ's, the latter having been already defined in Eq. (46).

Finally, the equilibrium specific heat is given by

$$\begin{aligned} C^{\text{Einstein}}_{(\omega_0\neq 0)} &= \frac{\partial E}{\partial T} = -2k_B - k_B\beta\frac{\hbar}{2\pi}\sum_{j=1}^{3}\left\{\frac{\lambda_j}{\nu}\psi'\left(\frac{\lambda_j}{\nu}\right)[2\omega_0^2 q_j + p_j] \right. \\ &\quad \left. + \frac{\lambda_j'}{\nu}\psi'\left(\frac{\lambda_j'}{\nu}\right)[2\omega_0^2 q_j' + p_j']\right\}, \end{aligned} \tag{71}$$

where $\psi'(z)$ are the trigamma functions.

We may now discuss the low and the high-temperature limits of Eq. (71).

(a) *Low-T limit.*

Employing the asymptotic expansion of the digamma function,

$$\psi'(z) = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^5} - \cdots, \tag{72}$$

we find

$$C^{\text{Einstein}}_{\omega\neq 0} = \frac{2\pi}{3}\frac{\gamma}{\omega_0^2}\frac{k_B^2 T}{\hbar} + \alpha_1^E T^3 + O(T^5). \tag{73}$$

where

$$\alpha_1^E = \frac{8\pi^3}{15}\frac{\gamma}{\omega_0}\frac{k_B^4}{(\hbar\omega_0)^3} \times\left\{\frac{3(\omega_c^2+\omega_0^2)}{\omega_0^2} - \left(\frac{\gamma}{\omega_0}\right)^2 - \frac{\omega_0}{\omega_D}\left(\frac{\omega_c^2}{\gamma\omega_0} - \frac{2\gamma}{\omega_0} - \frac{\omega_0}{\omega_D}\right)\right\}$$

As required by the third law of thermodynamics the specific heat does vanish as a power law as $T\to 0$, exactly in the same manner as in the corresponding Gibbs expression (cf., Eq. (48)), but interestingly, the coefficient of the next higher order term ($\propto T^3$) differs from the Gibbs result.

(b) *High-T limit.*

At high temperatures,

$$C^{\text{Einstein}}_{\omega\neq 0} = 2k_B - \frac{\alpha_2^E}{T^2}. \tag{74}$$

where

$$\alpha_2^E = \frac{\hbar^2}{12k_B}(\omega_c^2 + 2\omega_0^2 + \gamma\omega_D).$$

At infinite temperatures the classical equipartition result is restored. But again, in the next higher order term (in $1/T^2$), the Einstein result differs from the Gibbs result by a cutoff-dependent term:

$$C^{\text{Einstein}}_{\omega\neq 0} = C^{\text{Gibbs}}_{\omega\neq 0} + \frac{\hbar^2\gamma\omega_D}{12k_BT^2}. \tag{75}$$

D. Einstein approach ($\omega_0 \to 0$)

We now return to discuss the Einstein result for the specific heat due to the presence of the magnetic field alone, i.e.,

TABLE I. Comparison of specific heat in the Gibbs approach and the Einstein approach in different limits. ⋆ The expressions for the unconfined system in the Gibbs approach was obtained by starting with the oscillator potential absent in the Hamiltonian, whereas the expression for the Einstein approach was obtained by taking the limit of the confinement frequency ω_0 approaching zero before taking the $t \to \infty$ limit.

	$\omega_c \neq 0,\ \omega_0 \neq 0$		$\omega_c \neq 0$, unconfined system ⋆		$\omega_c = 0$, unconfined system ⋆	
	Low temperature	High temperature	Low temperature	High temperature	Low temperature	High temperature
Gibbs approach	$\frac{2\pi}{3}\frac{\gamma}{\omega_0^2}\frac{k_B^2 T}{\hbar} - \alpha_1^G T^3 + O(T^5)$	$2k_B - \frac{\alpha_2^G}{T^2}$	$\frac{2\pi}{3}\frac{\gamma}{\hbar}\frac{(1-\frac{\gamma}{\omega_D})}{\gamma^2+\omega_c^2} k_B^2 T - (\alpha_3^G - \alpha_4^G) T^3 + O(T^5)$	$k_B - \frac{\alpha_2^G\vert_{\omega_0=0}}{T^2}$	$\frac{2\pi}{3}\frac{k_B^2 T}{\hbar\gamma}(1-\frac{\gamma}{\omega_D}) - (\alpha_3^G\vert_{\omega_c=0} - \alpha_4^G) T^3 + O(T^5)$	$k_B - \frac{\alpha_2^G\vert_{(\omega_0=0,\ \omega_c=0)}}{T^2}$
Einstein approach	$\frac{2\pi}{3}\frac{\gamma}{\omega_0^2}\frac{k_B^2 T}{\hbar} - \alpha_1^E T^3 + O(T^5)$	$2k_B - \frac{\alpha_2^E}{T^2}$	$\frac{2\pi}{3}\frac{\gamma}{\hbar}\frac{1}{\gamma^2+\omega_c^2} k_B^2 T - \alpha_3^E T^3 + O(T^5)$	$k_B - \frac{\alpha_2^E\vert_{\omega_0=0}}{T^2}$	$\frac{2\pi}{3}\frac{k_B^2 T}{\hbar\gamma} - \alpha_3^E\vert_{\omega_c=0} T^3 + O(T^5)$	$k_B - \frac{\alpha_2^E\vert(\omega_c=0,\ \omega_0=0)}{T^2}$

in the absence of the parabolic well. The relevant Hamiltonian is

$$\mathcal{H}_S^{eff} = -\frac{1}{2}\hbar\omega_c + \frac{1}{2} m \dot{z}\dot{z}^{\dagger}, \qquad (76)$$

and hence

$$E = -\frac{1}{2}\hbar\omega_c + \frac{1}{2} m \lim_{t\to\infty}[\zeta_2(t)]_{\omega_0\to 0}. \qquad (77)$$

As discussed in Ref. [7], one of the three roots, viz. λ_1 vanishes as ω_0^2 for $\omega_0 \to 0$. Consequently, one obtains the internal energy by taking the limits carefully, (see Appendix B, for details),

$$E(\omega_0 \to 0) = k_B T + \frac{\hbar}{2\pi}\left\{ p_2 \psi\left(1 + \frac{\lambda_2}{\nu}\right) + p_3 \psi\left(1 + \frac{\lambda_3}{\nu}\right) + p_2' \psi\left(1 + \frac{\lambda_2'}{\nu}\right) + p_3' \psi\left(1 + \frac{\lambda_3'}{\nu}\right) \right\}. \qquad (78)$$

As before, the derivative of E with respect to temperature yields an expression for the specific heat in terms of the digamma functions, which can be further analyzed in the low- and high-temperature limits.

(a) *Low-T limit*.

Again, using the asymptotic expansion of the digamma function (cf., Eq. (72)), we find

$$C_{\omega\to 0}^{\text{Einstein}} = \frac{2\pi}{3}\frac{\gamma}{\hbar}\frac{1}{\gamma^2+\omega_c^2} k_B^2 T - \alpha_3^E T^3 + O(T^5). \qquad (79)$$

where

$$\alpha_3^E = \frac{8\pi^3}{15}\frac{k_B^4}{\hbar^3\sqrt{(\gamma^2+\omega_c^2)^3}}\left\{ \frac{(\gamma^3 - 3\gamma\omega_c^2)}{\sqrt{(\gamma^2+\omega_c^2)^3}}\left(1 - \frac{2\gamma}{\omega_D} - \left(\frac{\omega_c}{\omega_D}\right)^2\right) + 10\left(\frac{\omega_c}{\omega_D}\right)^2 \frac{\gamma(\gamma^2+\omega_c^2)}{\sqrt{(\gamma^2+\omega_c^2)^3}} \right\}$$

While the expression in Eq. (79) is in conformity with the third law of thermodynamics, as expected, it differs from the corresponding Gibbsian result of Eq. (53) in terms of different dependencies on the Drude cutoff ω_D. Apart from this issue the strong and weak magnetic field cases follow the behavior discussed earlier, below Eq. (53).

(b) *High-T limit*

$$C_{\omega_0\to 0}^{\text{Einstein}} = k_B - \frac{\hbar^2}{12 k_B T^2}(\omega_c^2 + \gamma\omega_D). \qquad (80)$$

Finally, in the high-temperature limit, equipartition result obtains, but once again, there is a correction term over and above the Gibbs result that is cutoff dependent, as we found earlier in the $\omega_0 \neq 0$ case in Eq. (75),

$$C_{\omega_0\to 0}^{\text{Einstein}} = C_F^{\text{Gibbs}} + \frac{\hbar^2\gamma\omega_D}{12 k_B T^2}, \qquad (81)$$

where $C_{\omega_0=0}^{\text{Gibbs}}$ is given by the high-T expression in Eq. (57).

V. SUMMARY

Summarizing, we study the various limiting behavior of the specific heat of a dissipative charged harmonic oscillator in a uniform magnetic field, obtained from the partition func-

TABLE II. Specific heat and magnetization in the limit of vanishing confinement frequency in two sequences.

	Specific heat		Magnetization
	Low temperature	High temperature	$(\gamma \to 0)$
$\omega_0 \to 0,\ t \to \infty$	$\frac{2\pi}{3}\frac{\gamma}{\hbar}\frac{1}{\gamma^2+\omega_c^2} k_B^2 T - \alpha_3^E T^3 + O(T^5)$	$k_B - \frac{\alpha_2^E\vert\omega_0=0}{T^2}$	$-\frac{\vert e\vert\hbar}{2mc}\coth(\frac{\hbar\omega_c}{2k_B T})$
$t \to \infty,\ \omega_0 \to 0$	Singularity	$2k_B - \frac{\alpha_2^E\vert_{\omega_0=0}}{T^2}$	$\frac{\vert e\vert\hbar}{2mc}[\frac{2k_B T}{\hbar\omega_c} - \coth(\frac{\hbar\omega_c}{2k_B T})]$

tion approach (Gibbs' method) and from the steady state of corresponding quantum Langevin equation (Einstein's approach). The specific heat obtained from both these methods shows linear T dependence at low temperatures, which is in agreement with the third law of thermodynamics. At high temperatures the specific heat approaches a constant value depending on the number of degrees of freedom of the system. Although, both the Gibbs and Einstein approaches are in conformity with the third law of thermodynamics and the equipartition theorem, at low and high temperatures respectively, they differ from each other in detail, beyond the leading order. In the limit of vanishing confinement frequency ($\omega_0 \to 0$), the specific heat of the oscillator becomes singular at low temperatures and manifests extra degrees of freedom counting at high temperatures. The specific heat of the free particle cannot be obtained from the equilibrium value ($t\to\infty$) of the specific heat of the oscillator just by taking the $\omega_0 \to 0$ limit. It is evident that the order in which one takes the $t=\infty$ and $\omega_0=0$ limits yield qualitatively different answers for the specific heat. While in the Einstein approach, the free particlelike specific heat emerges by taking the $\omega_0=0$ limit first before considering the $t=\infty$ limit, the Gibbs approach is plagued by a singularity issue, for $\omega_0=0$, in the low-temperature limit [cf., Eq. (48)]. The same issues have been pointed out in Ref. [4] earlier.

In Table I, we summarize our results for the Specific Heat in different limits. In the limit of $\omega_D \to \infty$, both the Gibbs and Einstein approaches give the same thermodynamic results. However, for a finite cutoff frequency ω_D, the results differ in next to the leading order at both high and low temperatures. The results summarized in Table I lead to the following conclusions:

(1) at low temperatures the specific heat is linear in temperature and hence the dissipative environment restores the third law of thermodynamics;

(2) in the presence of the oscillator potential, the low-temperature behavior of the specific heat goes as $1/\omega_0^2$ and is therefore singular in the limit of $\omega_0 \to 0$. Thus the results of the unconfined particle cannot be recovered in this limit.

(3) The high-temperature specific heat approaches a constant value independent of the confinement potential and depends only on the number of degrees of freedom in agreement with the equipartition law. Again, the results of the unconfined system cannot be recovered in the limit of vanishing confinement frequency ω_0.

While the issue of recovering the results of the unconfined particle, starting from the confined system and taking the limit of vanishing confinement frequency ω_0 cannot be resolved at the equilibrium level, the Einstein approach has the intrinsic advantage of obtaining the results in the process of equilibration. The equilibrium results can be arrived at by taking the limit of $t\to\infty$. Hence, one could in principle ask the question, what would happen if the confinement frequency ω_0 is taken to zero, before the limit $t\to\infty$ is taken. A similar result was obtained for the case of a particle in a harmonic-oscillator potential [22]. The results for the two different sequences of taking the limits are summarized in Table II. It is clear from the table that, if the limit of $\omega_0 \to 0$ is taken before the limit of $t\to\infty$, one can actually recover the results of the unconfined system for the specific heat and magnetization. It is curious to note that the result for magnetization obtained from this sequence of taking the limits is inconsistent with the Landau result, whereas when the limits are taken in the other way round, the Landau result is recovered. This is, however, due to the fact that the Landau result for magnetization can only be recovered in the presence of a confinement potential.

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APPENDIX A: EINSTEIN APPROACH ($\omega_0 \neq 0$)

With the help of the Drude cutoff frequency we can write $\chi(\omega)$ as

$$\chi(\omega) = \frac{(\omega_D - i\omega)}{[i\omega^3 - \omega^2(\omega_D + i\omega_c) - i\omega(\gamma\omega_D + i\omega_c\omega_D + \omega_0^2) + \omega_0^2\omega_D]} \tag{A1}$$

Alternatively,

$$\chi(\omega) = -\frac{(\omega + i\omega_D)}{(\omega + i\lambda_1)(\omega + i\lambda_2)(\omega + i\lambda_3)}, \tag{A2}$$

where $\lambda_j s$ and $\lambda_j' s$ are given by the Vieta equations [Eq. (46)]. We can write Eq. (66) as

$$\begin{aligned}\zeta_1(t,t') &= \langle z(t) z^\dagger(t')\rangle \\ &= \frac{1}{4\pi^2 m^2}\int_{-\infty}^{+\infty} d\tilde{\omega} f(\tilde{\omega}) \int_{-\infty}^{+\infty} d\omega \chi(\omega) \frac{(e^{-i\tilde{\omega}t} - e^{-i\omega t})}{i(\omega - \tilde{\omega})} \\ &\quad \times \int_{-\infty}^{+\infty} d\omega' \chi^*(\omega') \frac{(e^{i\tilde{\omega}t'} - e^{i\omega' t'})}{-i(\omega' - \tilde{\omega})}.\end{aligned} \tag{A3}$$

The two integrals, defined by

$$I_1 = \int_{-\infty}^{+\infty} d\omega \chi(\omega) \frac{(e^{-i\tilde{\omega}t} - e^{-i\omega t})}{i(\omega - \tilde{\omega})}, \quad \text{(A4)}$$

$$I_2 = \int_{-\infty}^{+\infty} d\omega' \chi^*(\omega') \frac{(e^{i\tilde{\omega}t'} - e^{i\omega' t'})}{-i(\omega' - \tilde{\omega})}, \quad \text{(A5)}$$

can be expressed as

$$I_1 = \frac{2\pi}{iA}\Bigg\{ \frac{(\lambda_1 - \omega_D)(\lambda_2 - \lambda_3)(e^{-i\tilde{\omega}t} - e^{-\lambda_1 t})}{(\tilde{\omega} + i\lambda_1)} + \frac{(\lambda_2 - \omega_D)(\lambda_3 - \lambda_1)(e^{-i\tilde{\omega}t} - e^{-\lambda_2 t})}{(\tilde{\omega} + i\lambda_2)} + \frac{(\lambda_3 - \omega_D)(\lambda_1 - \lambda_2)(e^{-i\tilde{\omega}t} - e^{-\lambda_3 t})}{(\tilde{\omega} + i\lambda_3)} \Bigg\}, \quad \text{(A6)}$$

$$I_2 = -\frac{2\pi}{iA'}\Bigg\{ \frac{(\lambda_1' - \omega_D)(\lambda_2' - \lambda_3')(e^{i\tilde{\omega}t'} - e^{-\lambda_1' t'})}{(\tilde{\omega} - i\lambda_1')} + \frac{(\lambda_2' - \omega_D)(\lambda_3' - \lambda_1')(e^{i\tilde{\omega}t'} - e^{-\lambda_2' t'})}{(\tilde{\omega} - i\lambda_2')} + \frac{(\lambda_3' - \omega_D)(\lambda_1' - \lambda_2')(e^{i\tilde{\omega}t'} - e^{-\lambda_3' t'})}{(\tilde{\omega} - i\lambda_3')} \Bigg\}. \quad \text{(A7)}$$

where

$$A = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3), \quad \text{(A8)}$$

$$A' = (\lambda_1' - \lambda_2')(\lambda_1' - \lambda_3')(\lambda_2' - \lambda_3'). \quad \text{(A9)}$$

Eq. (A3) then yields

$$\zeta_1(t,t') = \frac{1}{4\pi^2 m^2}\int_{-\infty}^{+\infty} d\tilde{\omega} f(\tilde{\omega}) I_1 I_2 = \frac{1}{4\pi^2 m^2}\int_{-\infty}^{+\infty} d\tilde{\omega}\frac{m}{\pi}\frac{\gamma\omega_D^2}{(\omega_D^2 + \tilde{\omega}^2)}\hbar\tilde{\omega}\coth\left(\frac{\hbar\tilde{\omega}}{2kT}\right) I_1 I_2 - \frac{1}{4\pi^2 m^2}\int_{-\infty}^{+\infty} d\tilde{\omega}\frac{m}{\pi}\frac{\gamma\omega_D^2}{(\omega_D^2 + \tilde{\omega}^2)}\hbar\tilde{\omega} I_1 I_2. \quad \text{(A10)}$$

The second integral in Eq. (A10) vanishes for symmetry reasons, so that only the integral containing cotangent hyperbolic contributes. In order to find out the equal-time-correlation function $\zeta_1(t)$, we set $t=t'$ in Eqs. (A6) and (A7). Also in the limit of $t\to\infty$, we can ignore the exponentials containing $\lambda_1, \lambda_2, \lambda_3$ and also the corresponding primed roots, and we are left with the terms whose exponentials contains $i\tilde{\omega}t$ and $-i\tilde{\omega}t$ only. The product $I_1 I_2$ in the above integral now become a time independent one. Finally,

$$\zeta_1 = \frac{\hbar}{4\pi^3 m}\int_{-\infty}^{+\infty} d\tilde{\omega}\frac{\gamma\omega_D^2}{(\omega_D^2 + \tilde{\omega}^2)}\tilde{\omega}\coth\left(\frac{\hbar\tilde{\omega}}{2k_B T}\right) I_1' I_2'. \quad \text{(A11)}$$

where

$$I_1' = \frac{2\pi}{i}\frac{(\omega_D - i\tilde{\omega})}{(\tilde{\omega} + i\lambda_1)(\tilde{\omega} + i\lambda_2)(\tilde{\omega} + i\lambda_3)}, \quad \text{(A12)}$$

$$I_2' = -\frac{2\pi}{i}\frac{(\omega_D + i\tilde{\omega})}{(\tilde{\omega} - i\lambda_1')(\tilde{\omega} - i\lambda_2')(\tilde{\omega} - i\lambda_3')}. \quad \text{(A13)}$$

We can write

$$\zeta_1 = Q_1 - Q_2. \quad \text{(A14)}$$

where

$$Q_1 = -\frac{\hbar}{2\pi m}\int_{-\infty}^{+\infty} d\tilde{\omega}\coth\left(\frac{\hbar\tilde{\omega}}{2kT}\right)\frac{(\omega_D - i\tilde{\omega})}{(\tilde{\omega} + i\lambda_1)(\tilde{\omega} + i\lambda_2)(\tilde{\omega} + i\lambda_3)},$$

$$Q_2 = \frac{\hbar}{2\pi m}\int_{-\infty}^{+\infty} d\tilde{\omega}\coth\left(\frac{\hbar\tilde{\omega}}{2kT}\right)\frac{(\omega_D + i\tilde{\omega})}{(\tilde{\omega} - i\lambda_1')(\tilde{\omega} - i\lambda_2')(\tilde{\omega} - i\lambda_3')}. \quad \text{(A15)}$$

After simplifications

$$\zeta_1 = \frac{2kT}{m\omega_0^2} + \frac{\hbar}{m\pi}\sum_{j=1}^{3}\left\{ q_j\psi\left(1 + \frac{\lambda_j}{\nu}\right) + q_j'\psi\left(1 + \frac{\lambda_j'}{\nu}\right)\right\}. \quad \text{(A16)}$$

where $\psi(1+z_j)$ is a digamma function, $\nu=\frac{2\pi kT}{\hbar}$, and the q_j and the q_j' are defined in Eq. (70). We can observe from Eq. (A16) (since $\langle \vec{r}^2\rangle = \langle zz^\dagger\rangle$) that the equipartition theorem is satisfied for this two-dimensional problem.

The equal-time-correlation function $\zeta_2(t)$, given in Eq. (61b), can also be calculated in a similar manner and in the limit of $t\to\infty$, we get,

$$\zeta_2 = \frac{2kT}{m} + \frac{\hbar\omega_0^2}{m\pi}\sum_{j=1}^{3}\left\{ q_j\psi\left(1 + \frac{\lambda_j}{\nu}\right) + q_j'\psi\left(1 + \frac{\lambda_j'}{\nu}\right)\right\} - \frac{\hbar}{m\pi}\left\{\sum_{j=1}^{3} p_j \sum_{n=1}^{\infty}\frac{1}{n + \frac{\lambda_j}{\nu}} + \sum_{j=1}^{3} p_j' \sum_{n=1}^{\infty}\frac{1}{n + \frac{\lambda_j'}{\nu}}\right\} + \frac{\hbar\omega_c}{m}, \quad \text{(A17)}$$

where q_j and the q_j' are defined in Eq. (70a), and p_j and p_j' are given by Eq. (70b). We now use a transformation $P_j = p_j + i\omega_c/3$, such that $\sum_{j=1}^{3} P_j = 0$, since $\sum_{j=1}^{3} p_j = -i\omega_c$. Therefore

$$\sum_{j=1}^{3} p_j \sum_{n=1}^{\infty}\frac{1}{n + \frac{\lambda_j}{\nu}} = \sum_{j=1}^{3} P_j \sum_{n=1}^{\infty}\frac{1}{n + \frac{\lambda_j}{\nu}} - \sum_{j=1}^{3} i\frac{\omega_c}{3}\sum_{n=1}^{\infty}\frac{1}{n + \frac{\lambda_j}{\nu}} = -\sum_{j=1}^{3} P_j\psi\left(1 + \frac{\lambda_j}{\nu}\right) - \sum_{j=1}^{3} i\frac{\omega_c}{3}\sum_{n=1}^{\infty}\frac{1}{n + \frac{\lambda_j}{\nu}}. \quad \text{(A18)}$$

In a similar fashion we can use a transformation $P_j' = p_j' - i\omega_c/3$, in such a way that $\sum_{j=1}^{3} P_j' = 0$ since $\sum_{j=1}^{3} p_j' = i\omega_c$, hence

$$\sum_{j=1}^{3} p_j' \sum_{n=1}^{\infty} \frac{1}{n+\frac{\lambda_j'}{\nu}} = \sum_{j=1}^{3} P_j' \sum_{n=1}^{\infty} \frac{1}{n+\frac{\lambda_j'}{\nu}} + \sum_{j=1}^{3} i\frac{\omega_c}{3} \sum_{n=1}^{\infty} \frac{1}{n+\frac{\lambda_j'}{\nu}}$$
$$= -\sum_{j=1}^{3} P_j' \psi\left(1+\frac{\lambda_j'}{\nu}\right) + \sum_{j=1}^{3} i\frac{\omega_c}{3} \sum_{n=1}^{\infty} \frac{1}{n+\frac{\lambda_j'}{\nu}}. \tag{A19}$$

Substituting Eqs. (A18) and (A19) in Eq. (A17) and using three important properties of the digamma functions [23]

$$\psi(x) - \psi(y) = \frac{(x-y)}{xy} + \sum_{n=1}^{\infty} \left[\frac{1}{n+y} - \frac{1}{n+x}\right],$$

$$\psi(1+z) = \psi(z) + \frac{1}{z},$$

$$\sum_{j=1}^{N} a_j \sum_{n=1}^{\infty} \frac{1}{n+z_j} = -\sum_{j=1}^{N} a_j \psi(1+z_j) \quad \left[\sum_{j=1}^{N} a_j = 0\right], \tag{A20}$$

we obtain

$$\zeta_2 = \frac{2kT}{m} + \frac{\hbar\omega_0^2}{m\pi} \sum_{j=1}^{3} \left\{ q_j \psi\left(1+\frac{\lambda_j}{\nu}\right) + q_j' \psi\left(1+\frac{\lambda_j'}{\nu}\right)\right\} + \frac{\hbar}{m\pi} \sum_{j=1}^{3} \left\{ p_j \psi\left(1+\frac{\lambda_j}{\nu}\right) + p_j' \psi\left(1+\frac{\lambda_j'}{\nu}\right)\right\} + \frac{\hbar\omega_c}{m}. \tag{A21}$$

From Eq. (A21), we can calculate the mean squared average of the kinematic momentum of the particle in a magnetic field, given by

$$\left\langle \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 \right\rangle = m^2\langle \dot{z}\dot{z}^\dagger\rangle - m\hbar\omega_c$$
$$= 2mkT + \frac{m\hbar\omega_0^2}{\pi} \times \sum_{j=1}^{3} \left\{ q_j \psi\left(1+\frac{\lambda_j}{\nu}\right) + q_j' \psi\left(1+\frac{\lambda_j'}{\nu}\right)\right\} + \frac{m\hbar}{\pi} \sum_{j=1}^{3} \left\{ p_j \psi\left(1+\frac{\lambda_j}{\nu}\right) + p_j' \psi\left(1+\frac{\lambda_j'}{\nu}\right)\right\}. \tag{A22}$$

In the limit of a vanishing magnetic field, the two average values which we calculate are similar to the result obtained for a damped harmonic oscillator, as given by Weiss [1], of course in two dimensions.

The internal energy can be obtained as

$$E(\omega_0) = \langle \mathcal{H}_S^{\text{eff}} \rangle = \frac{1}{2}m\langle \dot{z}\dot{z}^\dagger\rangle - \frac{1}{2}\hbar\omega_c + \frac{1}{2}m\omega_0^2\langle zz^\dagger\rangle. \tag{A23}$$

Taking the derivative with respect to temperature, we find

$$C_{\omega_0\neq 0}^{\text{Einstein}} = 2k_B - k_B\beta\frac{\hbar\omega_0^2}{\pi} \sum_{j=1}^{3} \left\{ q_j \frac{\lambda_j}{\nu} \psi'\left(1+\frac{\lambda_j}{\nu}\right) + q_j' \frac{\lambda_j'}{\nu} \psi'\left(1+\frac{\lambda_j'}{\nu}\right)\right\} - k_B\beta\frac{\hbar}{2\pi} \sum_{j=1}^{3} \left\{ p_j \frac{\lambda_j}{\nu} \psi'\left(1+\frac{\lambda_j}{\nu}\right) + p_j' \frac{\lambda_j'}{\nu} \psi'\left(1+\frac{\lambda_j'}{\nu}\right)\right\}, \tag{A24}$$

where $\psi'(z)$ are the trigamma functions and k_B is the Boltzmann constant. Finally employing the recurrence formula for trigamma functions leads to

$$\psi'(1+z) = \psi'(z) - \frac{1}{z^2},$$

and also

$$\sum_{j=1}^{3} \left\{ \frac{p_j}{\lambda_j} + \frac{p_j'}{\lambda_j'} \right\} = 0,$$

$$\sum_{j=1}^{3} \left\{ \frac{q_j}{\lambda_j} + \frac{q_j'}{\lambda_j'} \right\} = -\frac{1}{\omega_0^2}, \tag{A25}$$

from which we obtain Eq. (71).

APPENDIX B: EINSTEIN APPROACH ($\omega_0 \to 0$)

In this part we provide details of the calculations for the case of $\omega_0 \to 0$. From the Vieta equations given in Eq. (46), we can write the new equations for this particular case as $\lambda_2+\lambda_3=\omega_D+i\omega_c$, $\lambda_2\lambda_3=\omega_D(\gamma+i\omega_c)$ and λ_1 vanishes as ω_0^2. We started from Eq. (A3), keeping all the exponentials of Eqs. (A6) and (A7), set $t=t'$, in order to obtain the equal-time-correlation functions $\zeta_1(t)$ and $\zeta_2(t)$ in the vanishing oscillator frequency limit. Since λ_2 and λ_3 are of the order of γ, and in the limit $\gamma t \gg 1$, we can still ignore the exponential containing λ_2 and λ_3 and the corresponding primed roots. But as $\omega_0 \to 0$, λ_1 and $\lambda_1' \approx \omega_0^2$, and hence the exponentials containing λ_1 and λ_1' have to be treated carefully. The extra contribution due to these terms exactly cancels the singularity present in Eq. (A16) (as $\omega_0 \to 0$). In the limit of vanishing harmonic-oscillator frequency, the energy is obtained as Eq. (77),

$$E = -\frac{1}{2}\hbar\omega_c + \frac{1}{2}m \lim_{t\to\infty}[\zeta_2(t)]_{\omega_0\to 0}. \tag{B1}$$

We obtain

$$\lim_{t\to\infty}[\zeta_2(t)]_{\omega_0\to 0} = \frac{2k_BT}{m} + \frac{\hbar\omega_c}{m} + \frac{\hbar}{m\pi}\left\{p_2\psi\left(1+\frac{\lambda_2}{\nu}\right) + p_3\psi\left(1+\frac{\lambda_3}{\nu}\right) + p_2'\psi\left(1+\frac{\lambda_2'}{\nu}\right) + p_3'\psi\left(1+\frac{\lambda_3'}{\nu}\right)\right\}. \tag{B2}$$

where

$$p_2 = \frac{[\gamma\omega_D - i\omega_c(\lambda_2-\omega_D)]}{(\lambda_2-\lambda_3)}$$

$$p_3 = -\frac{[\gamma\omega_D - i\omega_c(\lambda_3-\omega_D)]}{(\lambda_2-\lambda_3)}. \tag{B3}$$

The primed roots are calculated from complex conjugates. Hence, the internal energy is

$$E(\omega_0\to 0) = k_BT + \frac{\hbar}{2\pi}\left\{p_2\psi\left(1+\frac{\lambda_2}{\nu}\right) + p_3\psi\left(1+\frac{\lambda_3}{\nu}\right) + p_2'\psi\left(1+\frac{\lambda_2'}{\nu}\right) + p_3'\psi\left(1+\frac{\lambda_3'}{\nu}\right)\right\}. \tag{B4}$$

Correspondingly, the specific heat becomes

$$C^{\text{Einstein}}_{\omega_0=0} = -k_B - k_B\beta\frac{\hbar}{2\pi}\left\{p_2\frac{\lambda_2}{\nu}\psi'\left(\frac{\lambda_2}{\nu}\right) + p_3\frac{\lambda_3}{\nu}\psi'\left(\frac{\lambda_3}{\nu}\right) + p_2'\frac{\lambda_2'}{\nu}\psi'\left(\frac{\lambda_2'}{\nu}\right) + p_3'\frac{\lambda_3'}{\nu}\psi'\left(\frac{\lambda_3'}{\nu}\right)\right\}. \tag{B5}$$

This form of the specific heat has been used in the text as the basis of our discussions of the low and high-temperature limits, via Eqs. (79) and (80).

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